PHYSICAL REVIEW E **66**, 015207(R) (2002)

Multistability, noise, and attractor hopping: The crucial role of chaotic saddles

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(Received 6 June 2001; revised manuscript received 3 May 2002; published 26 July 2002)

We investigate the hopping dynamics between different attractors in a multistable system under the influence of noise. Using symbolic dynamics we find a sudden increase of dynamical entropies, when a system parameter is varied. This effect is explained by a bifurcation involving two chaotic saddles. We also demonstrate that the transient lifetimes on the saddle obey a scaling law in analogy to crisis.

DOI: 10.1103/PhysRevE.66.015207 PACS number(s): 05.45.−a

Systems with a large number of coexisting stable states have been the subject of increasing interest recently. This multistable behavior occurs in many different fields such as optics [1], chemistry [2], neuroscience [3], semiconductor physics [4], plasma physics [5], and coupled oscillators [6]. Moreover, addition of noise to multistable systems has led to several interesting phenomena, such as noise-induced preference of attractors [7,8], directed diffusion [9], noise-enhanced multistability [10], and chaotic itinerancy [11]. The latter effect, which consists of hopping between different attractors caused by the noise, has also been observed experimentally in an optical system [12].

In this work we focus on the dynamics of the attractor-hopping process by taking a simple model as a paradigm. Due to the noise the attractors become metastable and the trajectory starts hopping between the different attractors. In the case of fractal basin boundaries, this hopping process consists of two steps. In the first one the trajectory leaves the open neighborhood about the attractor according to Arrhenius' law [13], in the second one, the trajectory bounces around on the chaotic saddle before cascading again into the open neighborhood of an attractor. The general mechanism for the first step of the hopping process is the same for bistable and multistable systems. In both cases we find Arrhenius' law; quanitative differences are due to different relative sizes of basins of attraction [8]. However, for the attractor-hopping dynamics, our study points out a major difference between bistable and multistable systems: While in bistable systems the structure of the saddle separating the two attractors (saddle point or chaotic saddle) does not play a role for the hopping characteristics, it is essential for the attractor hopping in a system, possessing a multitude of attractors. The structure of the chaotic saddles influences the nature of the hopping process, in particular, it determines which transitions between attractors are possible. Bifurcations in the chaotic saddles lead to changes in the hopping dynamics and thus to changes in the "accessibility" of attractors in the hopping process.

Employing symbolic dynamics, we assign one symbol to each attractor, thus transforming our hopping time series onto a symbolic string. As a next step we compute both the Shannon and the topological entropy, which can be regarded as measures of complexity for the hopping dynamics. As a system parameter is varied, the complexity of the hopping process changes beyond a certain threshold value. We show that this change can be related to a bifurcation, namely, a merging of two chaotic saddles accompanied by the emergence of additional points filling the gap between the formely separated saddles. This bifurcation is mediated by a snapback repellor [14], whose eigenvalues determine the scaling law for the transient lifetimes on one chaotic saddle close to the bifurcation point.

Our basic prototype model of multistablity, which captures the main features of highly multistable systems, namely, more than two final states (attractors) and a complexly interwoven fractal basin boundary separating these states, is given by the tenfold iterate of two coupled logistic maps,

$$x_{k+1}=1.0-\alpha x_k^2+\gamma(y_k-x_k),$$
$$y_{k+1}=1.0-\alpha y_k^2+\gamma(x_k-y_k). \tag{1}$$

We fix γ, the coupling strength, at $\gamma=0.29$, and vary α, the nonlinearity, in the range $\alpha\in[0.72,0.755]$. For this parameter set, there exists a stable period ten orbit for the two coupled logistic maps themselves. Since we consider the tenfold iterate, our map exhibits ten coexisting fixed point attractors, where each five lie above and below the symmetry axis $x=y$, respectively. Using only initial conditions below the symmetry axis, we can restrict our study to a system possessing five coexisting fixed point attractors, five repellors with two unstable directions, and ten saddle points. Each of the five attractors is surrounded by an open neighborhood of different size, in which all initial conditions converge to the corresponding attractor inside this neighborhood. The basins of attraction as a whole have a complex fractal structure due to a homoclinic bifurcation which occurs already for smaller values of the nonlinearity α. As a next step we apply noise to the system, using Gaussian white noise with standard deviation σ added to the x and y components of the tenfold iterate of Eq. (1). Small noise causes the system now to alternate between the different states, where long periods close to a fixed point, comparable to laminar motion, are interrupted by short, sudden bursts, which are reminiscent to intermittent behavior. During the bursts the motion takes place on the chaotic saddles separating the attractors, until the trajectory is again being injected in the neighborhood of one of the five fixed points. This complex dynamics is depicted in Fig. 1.

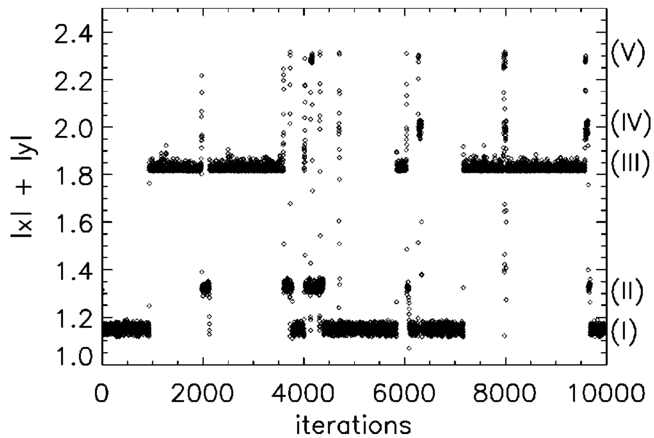

FIG. 1. Noisy time series of Eq. (1) for $\alpha=0.73$, $\gamma=0.29$, and $\sigma=0.012$. There are clearly five distinct almost periodic states, marked with roman numbers on the right-hand side. These states are interrupted by bursts, where the motion takes place on a chaotic saddle.

According to our aim, we are now interested in the characteristic properties of the hopping dynamics. Thus we address the question, how the complexity of the hopping dynamics changes with the nonlinearity parameter α, namely, whether every fixed point has a positive transition probability to every other fixed point, via a transient on a chaotic saddle. For this purpose, we employ the concept of symbolic dynamics. We use an encoding scheme in which we assign one symbol to each attractor [15]: Neglecting the number of iterations the trajectory spends close to a fixed point, a symbol is given for every fixed point and only after a jump out of an attractor a new symbol is bestowed, according to the attractor where the trajectory lands at. Using this scheme, we focus only on the structural properties of the jumps, not taking fully into account the complete temporal evolution in each iteration. As a quantitative measure of the complexity of the symbol string we use the Shannon entropy, in analogy to the Kolmogorov-Sinai entropy [16], given by

$$h_S=\lim_{n\to\infty}\frac{H_n}{n}=\lim_{n\to\infty}(H_{n+1}-H_n)$$
$$=\lim_{n\to\infty}\frac{1}{n}\left(-\sum_{|S|=n}p(S)\ln p(S)\right), \qquad (2)$$

where $S=s_1s_2\cdots s_n$ denotes a finite symbol sequence consisting of n elements $s_i=1,2,\ldots,5$, $p(S)$ its probability of occurrence, and H_n the block entropy of block length n. Numerically, the quantity $\lim_{n\to\infty}(H_{n+1}-H_n)$ converges already for $n=1$, indicating that the hopping dynamics is a Markov process of first order. This corresponds to the intuitive expectation, that the dynamics possesses only memory of the last state it was dwelling on. Higher correlations are suppressed by the long laminarlike motion. Consequently, a transition matrix between the different fixed points can be constructed, where the entries are the probabilities for a transition between two states.

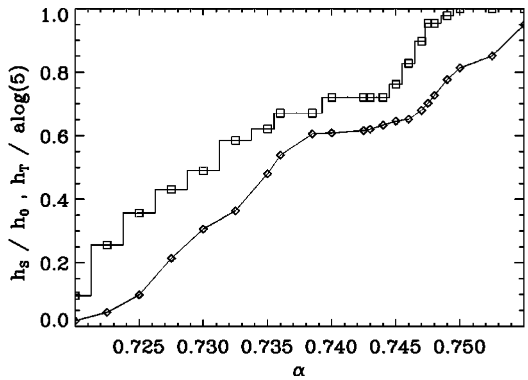

FIG. 2. The Shannon entropy (◇) and the topological entropy (□) of the symbol sequence generated from Eq. (1) under variation of the nonlinearity parameter α. Both quantities are normalized to 1.

Another interesting quantity to investigate is the topological entropy. One way of computing it is to build the Stefan transition matrix [17], where the entries of the 5×5 matrix are 1 and 0, depending on whether a transition took place or not, respectively. The logarithm of the largest eigenvalue of this matrix yields the topological entropy. In our numerical implementation, we use a certain, very small cutoff limit of 0.001 for the transition probabilities, below which we regard the value as zero. Thus we neglect transition, having extremely small probabilities compared to the other transitions. However, the exact value of this cutoff does not change the results significantly, as long as it is small enough. The entropies shown in Fig. 2 remain unchanged for cutoff limits between 10^{-3} and 10^{-6}. It is important to note that, although the *escape times* out of the stable states depend on the noise level, the *transition probabilities* remain constant for a wide range of noise values. Thus the considered entropies are not affected by the noise amplitude for a rather large interval of noise levels. In Fig. 2 we present the evolution of these two quantities as the nonlinearity α changes. The curve for the topological entropy appears to be a monotonic curve with a devil's staircase like behavior [18]. Every time it increases, at least one new transition between hitherto unconnected fixed points is created. Below the homoclinic bifurcation $\alpha<0.72$, where the basin boundaries are smooth, only one transition per fixed point is possible, as the fixed points are located on a closed curve made up by the unstable manifolds of the saddle points. These unstable manifolds impose a direction. Consequently, the topological entropy is zero. Above a value of $\alpha\approx0.75$, all transitions can occur and the maximum of the topological entropy is reached. The Shannon entropy is always less than the topological entropy. Besides the overall increase of both entropies, the plateau between $0.74<\alpha<0.744$ suggests that there is a change in the dynamical behavior beyond it. This change is due to a change in the topological structure of the system and relates to a new bifurcation as we show next. We compute the chaotic saddle, responsible for the chaotic dynamics, by the proper interior maximum triple method [19]. The result is depicted in Figs. 3 and 4, where the chaotic saddles are shown for two different values of α together with the fourfold preimages (+) of

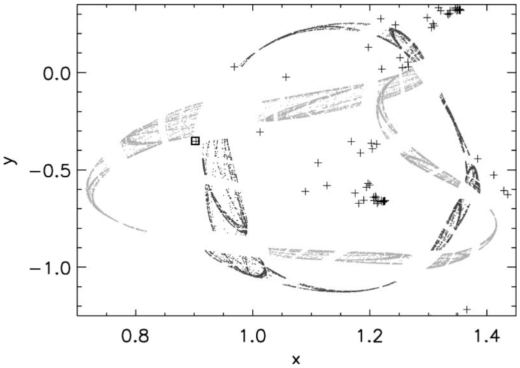

FIG. 3. Two chaotic saddle rings in different gray scale together with the fourfold preimages (+) of the repellor R marked with a □ for $\alpha = 0.740$.

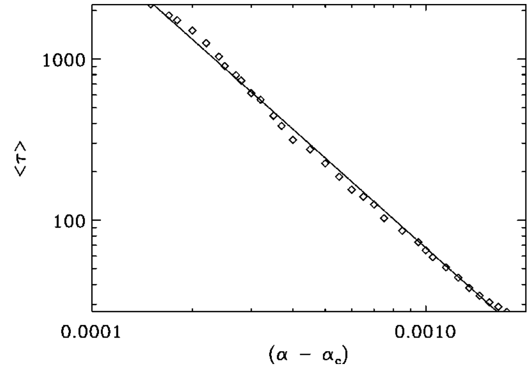

FIG. 5. Scaling of the transient lifetime to stay in one ring of the chaotic saddle with the nonlinearity α and $\alpha_c = 0.7428$. The slope of the log-log plot is $\epsilon = 1.86$.

one repellor R marked with □. Slightly above the homoclinic bifurcation ($\alpha = 0.725$) the saddles are very thin, consisting of two separate rings. At a value of $\alpha = 0.755$, there exists only one large piece of the nonattracting chaotic set (not shown). This results from the saddle merging, which took place at $\alpha \approx 0.743$ (Fig. 4). The two bands of chaotic saddles merge and the repellor, formerly located ouside of the saddles, is now embedded in the large saddle, including some of its preimages. The merging is accompanied by the successive filling of the formerly empty gap between the two saddles. A similar effect has been observed in Ref. [20] where it is called "spilling," in chaotic scattering [21], and in the gap filling crisis [22]. In contrast to the latter this happens smoothly in the saddle merging.

However, the actual bifurcation takes place at a slightly lower value of $\alpha \approx 0.7428$ (not shown). At this parameter value some of the preimages of the marked repellor touch its unstable manifold, which constitutes the border of the chaotic saddle. Through this mechanism, a snapback repellor has developed. A fixed point p is called a snapback repellor, if (i) all eigenvalues of p have absolute values larger than 1, (ii) there exists a $q \in W^u_{loc}(p)$, the unstable manifold of p, such that the M-fold iterate of the map $\mathcal{F}^M(q) = p$ for some positive integer M and (iii) $\det D\mathcal{F}^M(q) \neq 0$ [14]. While conditions (i) and (iii) are true for all α considered here, it is condition (ii) that becomes fulfilled during the bifurcation, as the unstable manifold of the repellor touches its preimages.

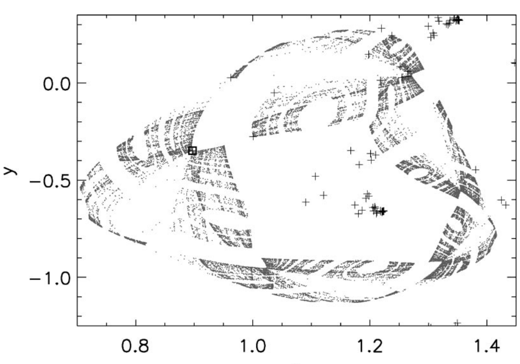

FIG. 4. Chaotic saddle together with the fourfold preimages (+) of the repellor R marked with a □ for $\alpha = 0.743$.

This bifurcation is responsible for the emergence of the additional points in the gaps, (see Fig. 4). At a slightly higher value of $\alpha \approx 0.743$, the preimages touch not only the unstable manifold, constituting the border of the chaotic saddle, but also the saddle itself. As the chaotic saddle is an invariant set, the repellor becomes thus embedded in the saddle and the merging of the two rings has taken place. This merging has an important consequence for the dynamics of the attractor hopping. Not only jumps between neighbors, but also arbitrary jumps become, in principle, possible. Above the critical value, the connected pieces of the chaotic saddle enable the trajectory to jump between the formerly separated parts, thereby causing a sharp increase in the number of the possible transitions, as can be seen from the topological entropy above $\alpha = 0.743$ in Fig. 2. Since the bifurcation described above can be considered as a kind of crisis, it is interesting to see whether the scaling laws for the characteristic transient lifetimes [23] apply in this case as well. We compute the average escape times out of one (still separated) chaotic saddle ring slightly below the bifurcation. In Fig. 5 the data are plotted, resulting in a scaling relation

$$\langle \tau \rangle \sim (\alpha - \alpha_c)^{-\epsilon}, \tag{3}$$

where $\alpha_c = 0.7428$. The best fit to the data yields an exponent $\epsilon = 1.86$. That value can be reproduced by

$$\epsilon = 1/2 + [\ln(\beta_1)/\ln(\beta_2)], \tag{4}$$

$$\epsilon = 1/2 + [\ln(2.45)/\ln(1.9)] = 1.85, \tag{5}$$

where β_1 and β_2 are the expanding eigenvalues of the mediating repellor.

This coincidence has been confirmed also for other values of the coupling strength $\gamma = 0.285$, 0.287, and 0.292. It is worth mentioning that in the original derivation of this formula, which applies to the heteroclinic tangency crisis, β_1 and β_2 are the expanding and contracting eigenvalues of the mediating saddle point [24]. However, in our case both β_1 and β_2 are expanding eigenvalues of the mediating repellor. We conjecture this relation to hold for the general case of a crisis caused by a snapback repellor. The reason that a law for the heteroclinic tangency is supposed to hold in a homoclinic case of a snapback repellor is that the mechanism for approaching the fixed point is different here. The fixed point is not reached by successive iterations, where each one is closer to the fixed point as the former one, but suddenly with a jump of noninfinitesimal size from a finite distance.

In conclusion, we have studied a simple paradigmatic map possessing a multitude of coexisting stable states under the influence of a small noise. We have shown that the nature of the attractor-hopping process depends, in contrast to the bistable case, crucially on the structure of the chaotic saddles separating the attractors. Bifurcations of the saddles lead to a change in the hopping dynamics which is manifested in a sudden increase of dynamical entropies. In particular, we described a bifurcation, a merging of two chaotic saddles involving a snapback repellor, which results in a change of the ‘‘accessibility’’ of the attractors. Finally, a conjecture for the scaling law of the transient lifetimes on the chaotic saddle has been provided. The hopping dynamics, the bifurcation, and the scaling law should be observable in experimental realizations of such systems. We expect this bifurcation also to occur in systems with three or more dimensions, where the mediating saddle has at least two unstable directions.

We acknowledge Y. Maistrenko for valuable discussions and the DFG and INTAS for financial support.

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